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Levinson's Theorem for the Schrödinger Equation in One Dimension

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Levinson's theorem for the one-dimensional Schrödinger equation with a symmetric potential which decays at infinity faster than x^{-2} is established by the Sturm–Liouville theorem. The critical case where the Schrödinger equation has a finite zero-energy solution is also analyzed. It is demonstrated that the number of bound states with even (odd) parity $n_+(n_-)$ is related to the phase shift $\eta_+(0)$ $[\eta_-(0)]$ of the scattering states with the same parity at zero momentum as $\eta_+(0) + \pi/2 = n_+\pi$ and $\eta_-(0) = n_-\pi$ for the noncritical case, and $\eta_+(0) = n_+\pi$ and $\eta_-(0) - \pi/2 = n_-\pi$ for the critical case.

1. INTRODUCTION

The Levinson theorem (Levinson, 1949) is an important theorem in nonrelativistic quantum scattering theory that establishes a relation between the total number n_l of bound states with angular momentum l and the phase shift $\delta_l(0)$ of the scattering state at zero momentum for the Schrödinger equation with a spherically symmetric potential $V(r)$ in three dimensions:

$$\delta_l(0) - \delta_l(\infty) = \begin{cases} (n_l + 1/2)\pi & \text{when } l = 0 \text{ and a half-bound state occurs} \\ n_l\pi & \text{the remaining cases} \end{cases} \tag{1}$$

where the potential $V(r)$ satisfies the following asymptotic conditions:

$$r^2|V(r)| \to 0 \qquad \text{at} \quad r \to 0 \tag{2}$$

$$r^3|V(r)| \to 0 \qquad \text{at} \quad r \to \infty \tag{3}$$

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These two conditions are necessary for the nice behavior of the wave function at the origin and the analytic property of the Jost function, respectively. The first line in Eq. (1) was first expressed by Newton (Newton, 1960, 1977a, b, 1982) for the case where a half-bound state of the *S* wave occurs. The zero-energy solution to the Schrödinger equation is called a half-bound state provided that its wave function is finite, but does not decay fast enough at infinity to be square-integrable.

During the past half century, the Levinson theorem has been proved by several authors with different methods and generalized to different fields (Levinson, 1949; Newton, 1960, 1977a, b, 1982, 1994; Jauch, 1957; Martin, 1958; Ni, 1979; Ma and Ni, 1985; Ma, 1985a, b, 1996; Iwinski *et al.*, 1985, 1986; Rosenberg and Spruch, 1996; Liang and Ma, 1986; Poliatzky, 1993; Blankenbecler *et al.*, 1986; Niemi and Semenoff, 1985; Vidal and LeTourneux, 1992; Kiers and van Dijk, 1996; de Bianchi, 1994; Martin and de Bianchi, 1996; Portnoi and Galbraith, 1997, 1998; Bollé *et al.*, 1986; Gibson, 1987). Most of this work mainly studied the Levinson theorem in three-dimensional space. With the recent wide interest in lower dimensional field theories, the two-dimensional Levinson theorem has been studied numerically (Portnoi and Galbraith, 1997) as well as theoretically (Portnoi and Galbraith, 1998; Bollé *et al.*, 1986; Gibson, 1987; Lin, 1997, 1998; Dong *et al.*, 1998a–c). With respect to the two-dimensional Schrödinger equation, the Levinson theorem can be given as

$$\eta_m(0) = \begin{cases} (n_m + 1)\pi & \text{when } m = 1 \text{ and a half-bound state occurs} \\ n_m\pi & \text{the remaining cases} \end{cases} \tag{4}$$

where $\eta_m(0)$ is the limit of the phase shifts at zero momentum for the *m*th partial wave, and n_m is the total number of bound states with the given angular momentum $m\hbar$.

Due to the wide interest in lower dimensional field theory, it may be worth also studying the Levinson theorem in one dimension for completeness. In fact, it is common knowledge that one-dimensional quantum scattering describes many actual physical phenomena to a good approximation. For instance, the problem of tunneling times has been discussed in Hauge and Støvneng (1989). Furthermore, one-dimensional models are often applied to make the more complex higher dimensional systems tractable. Consequently, it seems reasonable to study the one-dimensional Levinson theorem. This will be beneficial for understanding both the two-dimensional Levinson theorem and the three-dimensional theorem. Actually, it seems that the direct or implicit study of the one-dimensional Levinson theorem (de Bianchi, 1994; Jackiw and Woo, 1975; Newton, 1980, 1983, 1984; Baton, 1985; Kiers and van Dijk, 1996; Aktosun *et al.*, 1993, 1996, 1998a, b; Nogami and Ross, 1996; Eberly, 1965) has attracted much more attention than that of the two-dimensional theorem. We approach this problem by the Sturm–Liouville theorem (Yang, 1982)

Generally speaking, there are several methods for studying the one-dimensional Levinson theorem for a nonrelativistic particle. One is based on the partial-wave analysis method (Nogami and Ross, 1996; Eberly, 1965). A second relies on the parity-eigenstate representation method (de Bianchi, 1994; van Dijk and Thiers, 1992). A third is to establish the Levinson theorem by the Jost function and the *S*-matrix method (Baton, 1985), which is essentially based on the orthogonality and completeness relation for the eigenfunctions of the total Hamiltonian, as was first noticed by Jauch (1957).

The purpose of this paper is to demonstrate the one-dimensional Levinson theorem for the Schrödinger equation by the Sturm–Liouville theorem. We arrive at the final result

$$\begin{aligned} \eta_+(0) + \pi/2 &= n_+\pi, & \eta_-(0) &= n_-\pi & &\text{for the noncritical case} \\ \eta_+(0) &= n_+\pi, & \eta_-(0) - \pi/2 &= n_-\pi & &\text{for the critical case} \end{aligned} \tag{5}$$

where n_+ and n_- denote the number of bound states with even parity and odd parity, and $\eta_+(0)$ and $\eta_-(0)$ denote the phase shifts of the scattering states with the same parity at zero momentum, respectively. This conclusion coincides with that in de Bianchi (1994).

One can readily find from Eqs. (5) that the Levinson theorem for the odd-parity case in one dimension is the same as that for the case $l = 0$ in three dimensions. However, the even-parity case has no counterpart compared to the three-dimensional Levinson theorem. This is a very interesting feature.

This paper is organized as follows. For simplicity, we first discuss the cutoff potential case, where the potential is vanishing beyond a sufficiently large distance x_0, and leave to Section 5 the discussion for the general case where the potential has a tail at infinity. In Section 2 the logarithmic derivative of the wave function of the Schrödinger equation is chosen as the phase angle (Yang, 1982), which is proved to be monotonic with respect to the energy (the Sturm–Liouville theorem). In Section 3, according to this monotonic property, the number of bound states is shown to be related to the logarithmic derivative of zero energy at x_0 when the potential changes from zero to the given value. It will be further shown in Section 4 that the logarithmic derivative of zero energy at x_0 also determines the limit of the phase shifts at zero momentum, which leads to the establishment of the one-dimensional Levinson theorem. The critical case, where a zero-energy solution occurs, is also analyzed there.

2. NOTATIONS AND THE STURM–LIOUVILLE THEOREM

Throughout this paper the natural units $\hbar = 1$ and $2m = 1$ are employed. Let us consider the one-dimensional Schrödinger equation with a symmetric potential $V(x)$,

$$\frac{d^2\psi(x)}{dx^2} + [E - V(x)]\psi(x) = 0, \qquad V(-x) = V(x)$$

where E denotes the energy of the particle. For simplicity, we first discuss the case with a cutoff potential:

$$V(x) = 0 \qquad \text{when} \quad x \geq x_0 \tag{6}$$

where x_0 is a sufficiently large distance. Introduce a parameter λ for the potential $V(x)$:

$$V(x, \lambda) = \lambda V(x) \tag{7}$$

where the potential $V(x, \lambda)$ changes from zero to the given potential $V(x)$ as λ increases from zero to one. After introducing the parameter λ, the one-dimensional Schrödinger equation can be modified as

$$\frac{\partial^2}{\partial x^2}\psi(x, \lambda) + [E - V(x, \lambda)]\psi(x, \lambda) = 0 \tag{8}$$

Since the potential is symmetric, the energy eigenfunctions can be combined into those with a definite parity, which satisfy the following boundary conditions at the origin:

$$\begin{aligned} &\psi^{(o)}(x, \lambda)\big|_{x=0} = 0 \qquad \text{for the odd-parity case} \\ &\left.\frac{\partial\psi^{(e)}(x, \lambda)}{\partial x}\right|_{x=0} = 0 \qquad \text{for the even-parity case} \end{aligned} \tag{9}$$

Therefore, in the course of studying the one-dimensional Levinson theorem we only need to discuss the wavefunction in the range $0 \leq x < \infty$ with the given parities, in the even-parity and odd-parity cases, respectively.

Now, we are going to solve Eq. (8) in two ranges $[0, x_0]$ and $[x_0, \infty)$, and match two solutions at x_0. Ignoring the effect of the normalization factor, which is irrelevant to our discussion, we only need one matching condition at x_0, which is the condition for the logarithmic derivative of the wave function (Yang, 1982):

$$A(E, \lambda) \equiv \left\{\frac{1}{\psi(x, \lambda)}\frac{\partial\psi(x, \lambda)}{\partial x}\right\}_{x=x_0-} = \left\{\frac{1}{\psi(x, \lambda)}\frac{\partial\psi(x, \lambda)}{\partial x}\right\}_{x=x_0+} \tag{10}$$

According to the condition (9), there exists only one solution near the origin. For example, for the free particle ($\lambda = 0$), the solution to Eq. (8) at the range $[0, x_0]$ is real:

$$\psi^{(e)}(x, 0) = \begin{cases} \cos(kx) & \text{when} \quad E = k^2 > 0 \\ \cosh(\kappa x), & \text{when} \quad E = -\kappa^2 \leq 0 \end{cases} \tag{11}$$

for the even-parity case, and

$$\psi^{(o)}(x, 0) = \begin{cases} \sin(kx) & \text{when} \quad E = k^2 > 0 \\ \sinh(\kappa x), & \text{when} \quad E = -\kappa^2 \leq 0 \end{cases} \tag{12}$$

for the odd-parity case.

In the range (x_0, ∞), we have $V(x) = 0$. For $E > 0$, there exist two oscillatory solutions to Eq. (8) whose combination can always satisfy the matching condition (10), so that there is a continuous spectrum for $E > 0$. Assuming that the phase shifts $\eta_{\pm}(k, \lambda)$ are zero for the free particles ($\lambda = 0$), we have

$$\psi(x, \lambda) = \begin{cases} \cos(kx + \eta_{+}(k, \lambda)), & \text{for the even-parity case} \\ \sin(kx + \eta_{-}(k, \lambda)), & \text{for the odd-parity case} \end{cases} \tag{13}$$

$$\eta_{\pm}(k, 0) = 0, \qquad \text{when} \quad k > 0 \tag{14}$$

We would like to make some remarks here. First, at the first sight, the wavefunction in Eq. (13) seems not to have a definite parity. As a matter of fact, the solutions (13) are only suitable in the region (x_0, ∞). The corresponding solutions in the region $(-\infty, -x_0]$ can be calculated according to the parity of the solution. For example, in the odd-parity case, the solution in the region $(-\infty, -x_0)$ is

$$-\sin(k|x| + \eta_{-}(k, \lambda)) = \sin(kx - \eta_{-}(k, \lambda))$$

Second, the solution (13) for the even-parity case can be rewritten as

$$\sin(kx + \eta_{+}(k, \lambda) + \pi/2) \tag{15}$$

$\eta_{+}(k, \lambda) + \pi/2$ plays the same role in the even-parity case as $\eta_{-}(k, \lambda)$ in the odd-parity case. Therefore, we only need to establish the Levinson theorem for the odd-parity case, and the Levinson theorem for the even-parity case can be obtained by replacing $\eta_{-}(k, \lambda)$ with $\eta_{+}(k, \lambda) + \pi/2$.

Finally, in the region $[x_0, \infty)$, the potential $V(x, \lambda)$ is vanishing and does not depend on λ. However, the phase shifts $\eta_{\pm}(k, \lambda)$ depend on λ through the matching condition (10):

$$\tan\eta_-(k, \lambda) = -\tan(kx_0)\,\frac{A(E, \lambda) - k\cot(kx_0)}{A(E, \lambda) + k\tan(kx_0)} \tag{16}$$

for the odd-parity case, and a similar formula for the even-parity case can be obtained by replacing $\eta_-(k, \lambda)$ with $\eta_+(k, \lambda) + \pi/2$.

The phase shifts $\eta_-(k, \lambda)$ are determined from Eq. (16) up to a multiple of π due to the period of the tangent function. In our convention (14), the phase shift $\eta_-(k, \lambda)$, $k > 0$, changes continuously as λ increases from zero to one. In other words, the phase shift $\eta_-(k, \lambda)$ is determined completely in our convention, and so is $\eta_+(k, \lambda)$. For simplicity we define

$$\eta_\pm(k) \equiv \eta_\pm(k, 1) \tag{17}$$

Since there is only one finite solution at infinity for $E \leq 0$, both for the even-parity case and for the odd-parity case,

$$\psi(x, \lambda) = \exp(-\kappa x) \qquad \text{when} \quad x_0 \leq x < \infty \tag{18}$$

The solution satisfying the matching condition (10) will not always exist for $E \leq 0$. Except for $E = 0$, if and only if there exists a solution of energy E satisfying the matching condition (10) does a bound state appear at this energy. Therefore, there is a discrete spectrum for $E \leq 0$. The finite solution for $E = 0$ is a constant one. It does not decay fast enough to be square-integrable such that it is not a bound state if the matching condition (10) is satisfied.

We now turn to the Sturm–Liouville theorem. Denote by $\overline{\psi}(x, \lambda)$ the solution to Eq. (8) corresponding to the energy $\overline{E}$,

$$\frac{\partial^2}{\partial x^2}\overline{\psi}(x, \lambda) + [\overline{E} - V(x, \lambda)]\overline{\psi}(x, \lambda) = 0 \tag{19}$$

Multiplying Eq. (8) and Eq. (19) by $\overline{\psi}(x, \lambda)$ and $\psi(x, \lambda)$, respectively, and calculating their difference, we obtain

$$\frac{\partial}{\partial x}\left\{\psi(x, \lambda)\,\frac{\partial\overline{\psi}(x, \lambda)}{\partial x} - \overline{\psi}(x, \lambda)\,\frac{\partial\psi(x, \lambda)}{\partial x}\right\} = -(\overline{E} - E)\overline{\psi}(x, \lambda)\psi(x, \lambda) \tag{20}$$

According to the boundary condition (9), the derivatives of the wavefunction for the even-parity case and the wavefunction for the odd-parity case vanish at the origin, respectively. Therefore, integrating (20) in the range $0 \leq x \leq x_0$, we obtain

$$\frac{1}{\overline{E} - E}\left\{\psi(x, \lambda)\frac{\partial\overline{\psi}(x, \lambda)}{\partial x} - \overline{\psi}(x)\frac{\partial\psi(x, \lambda)}{\partial x}\right\}_{x=x_0-} = -\int_0^{x_0}\overline{\psi}(x, \lambda)\psi(x, \lambda)\, dx$$

Taking the limit, we arrive at

$$\frac{\partial A(E, \lambda)}{\partial E} = \frac{\partial}{\partial E}\left(\frac{1}{\psi(x, \lambda)}\frac{\partial\psi(x, \lambda)}{\partial x}\right)_{x=x_0-} = -\psi(x_0, \lambda)^{-2}\int_0^{x_0}\psi(x, \lambda)^2\, dx \leq 0 \tag{21}$$

Similarly, from the boundary conditions that when $E < 0$ the function $\psi(x, \lambda)$ tends to zero at infinity, and when $E = 0$ the derivative of the function is equal to to zero at infinity, we have

$$\frac{\partial}{\partial E}\left(\frac{1}{\psi(x, \lambda)}\frac{\partial\psi(x, \lambda)}{\partial x}\right)_{x=x_0+} = \psi(x_0, \lambda)^{-2}\int_{x_0}^{\infty}\psi(x, \lambda)^2\, dx > 0 \tag{22}$$

Therefore, when $E \leq 0$, it is evident that both sides of Eq. (10) are monotonic with respect to the energy E: as the energy increases, the logarithmic derivative of the wave function at x_0- decreases monotonically, but that at x_0+ increases monotonically. This is the essence of the Sturm–Liouville theorem.

3. THE NUMBER OF BOUND STATES

In this section we will establish the relation between the number of bound states and the logarithmic derivative $A(0, \lambda)$ of the wavefunction at $x = x_0-$ for zero energy when the potential changes, in terms of the monotonic property of the logarithmic derivative of the wave function with respect to the energy E.

For $E \leq 0$, we obtain the logarithmic derivative at $x = x_0+$ from Eq. (18):

$$\left(\frac{1}{\psi(x, \lambda)}\frac{\partial\psi(x, \lambda)}{\partial x}\right)_{x=x_0+} = \begin{cases} 0 & \text{when} \quad E \sim 0 \\ -\kappa \sim -\infty & \text{when} \quad E \to -\infty \end{cases} \tag{23}$$

On the other hand, when $\lambda = 0$, the logarithmic derivative at $x = x_0-$ can be calculated from Eqs. (11) and (12) for $E \leq 0$:

$$A(E, 0) = \left(\frac{1}{\psi(x, 0)}\frac{\partial\psi(x, 0)}{\partial x}\right)_{x=x_0-} = \kappa \tanh(\kappa x_0)$$

$$= \begin{cases} 0 & \text{when } E \sim 0 \\ \kappa \sim \infty & \text{when } E \to -\infty \end{cases} \tag{24}$$

for the even-parity case, and

$$A(E,0) = \left(\frac{1}{\psi(x,0)} \frac{\partial \psi(x,0)}{\partial x} \right)_{x=x_0-} = \kappa \coth(\kappa x_0)$$

$$= \begin{cases} x_0^{-1} & \text{when } E \sim 0 \\ \kappa \sim \infty & \text{when } E \to -\infty \end{cases} \tag{25}$$

for the odd-parity case.

It is evident from Eqs. (23) and (25) that there is no overlap between two variant ranges of two logarithmic derivatives for the odd-parity case, namely there is no bound state for the free particle in the odd-parity case. However, there is one point overlap from Eqs. (23) and (24). It means that there is a finite solution at $E = 0$ when $\lambda = 0$ for the even-parity case. It is nothing but a constant solution. This solution is finite, but does not decay fast enough at infinity to be square-integrable. It is not a bound state, and is called a half-bound state. We will discuss the cases with a half-bound state later.

Now, both for the even-parity case and for the odd-parity case, if $A(0, \lambda)$ decreases across the value zero as λ increases, an overlap between the variant ranges of two logarithmic derivatives of two sides of $x = x_0$ appears. Since the logarithmic derivative of the wavefunction at x_0- decreases monotonically as the energy increases and that at x_0+ increases monotonically, the overlap means that there must exist one and only one energy for which the matching condition (10) is satisfied, that is, a bound state appears. From the viewpoint of node theory, when $A(0, \lambda)$ decreases across the value zero, a node for the zero-energy solution to the Schrödinger equation comes inward from the infinity, namely, a scattering state changes to a bound state.

As λ increases again, $A(0, \lambda)$ can decrease to $-\infty$, jump to ∞, and then decrease again across the value zero, so that another overlap occurs and another bound state appears. Note that when the zero point in the zero-energy solution $\psi(x, \lambda)$ comes to $x = x_0$, $A(0, \lambda)$ goes to infinity. It is not a singularity.

Each time $A(0, \lambda)$ decreases across the value zero, a new overlap between the variant ranges of two logarithmic derivatives appears such that a scattering state changes to a bound state. At the same time, a new node comes inward from infinity in the zero-energy solution to the Schrödinger equation. Conversely, each time $A(0, \lambda)$ increases across the value zero, an overlap between those two variant ranges disappears so that a bound state changes back to a scattering state, and simultaneously, a node goes outward and disappears in the zero-energy solution. The number of bound states $n_\pm$ is equal to the times that $A(0, \lambda)$ decreases across the value zero as λ increases from zero to one, subtracted by the times that $A(0, \lambda)$ increases across the value zero. It is also equal to the number of nodes in the zero-energy solution. In the next section we will show that this number is nothing but the phase shift at zero momentum divided by π, i.e., $\eta_-(0)/\pi$ or $\eta_+(0)/\pi + 1/2$.

We should pay some attention to the critical case where $A(0, 1) = 0$. A finite zero-energy solution $\psi(x, 1) = c$ at $[x_0, \infty)$ will satisfy the matching condition (10) with the zero $A(0, 1)$. Note that when $A(0, 1) = 0$, the wavefunction at x_0-, $\psi(x_0, 1)$, must be nonvanishing for the nontrivial solution. The constant c is nothing but the nonvanishing value $\psi(x_0, 1)$. The constant solution is not square-integrable so that it is not a bound state, and is called a half-bound state. As λ increases from a number near and smaller than one and finally to reach one, if $A(0, \lambda)$ decreases and finally reaches the value zero, a scattering state becomes a half-bound state, and no new bound state appears. Conversely, as λ increases to reach one, if $A(0, \lambda)$ increases and finally reaches the value zero, a bound state becomes a half-bound state, namely, a bound state disappears. This conclusion holds for both the even-parity case and the odd-parity case.

4. LEVINSON'S THEOREM

When $\lambda = 0$, the phase shifts $\eta_\pm(k, 0)$ are defined to be zero. As λ increases from zero to one, $\eta_\pm(k, 0)$ for $k > 0$ changes continuously.

For the odd-parity case, the phase shift $\eta_-(k, \lambda)$ is calculated by Eq. (16). It is easy to see that the phase shift $\eta_\pm(k, \lambda)$ increases monotonically as the logarithmic derivative $A(E, \lambda)$ decreases:

$$\left.\frac{\partial \eta_-(k, \lambda)}{\partial A(E, \lambda)}\right|_k = \frac{-k \cos^2 \eta_-(k, \lambda)}{\{A \cos(kx) + k \sin(kx)\}^2} \leq 0 \tag{26}$$

The phase shift $\eta_-(0, \lambda)$ is the limit of the phase shift $\eta_-(k, \lambda)$ as k tends to zero. Therefore, we are only interested in the phase shift $\eta_-(k, \lambda)$ at a sufficiently small momentum k, $k \ll 1/x_0$. For the small momentum we obtain from Eq. (16)

$$\tan \eta_-(k, \lambda) \sim -(kx_0) \frac{A(0, \lambda) - c^2k^2 - x_0^{-1} + k^2 x_0/3}{A(0, \lambda) - c^2k^2 + k^2x_0} \tag{27}$$

where the expansion of $A(E, \lambda)$ for small k is used,

$$A(E, \lambda) \sim A(0, \lambda) - c^2k^2, \qquad c^2 \geq 0 \tag{28}$$

which is calculated from the Sturm–Liouville theorem (21). In both the numerator and the denominator of Eq. (27) we included the next leading term, which is only useful for the critical cases where the leading terms cancel each other.

First, it can be seen from Eq. (27) that, except for the special point where $A(0, \lambda) = 0$, $\tan \eta_-(k, \lambda)$ tends to zero as k goes to zero, namely,

$\eta_-(0, \lambda)$ is always equal to the multiple of π except for $A(0, \lambda) = 0$. In other words, if the phase shift $\eta_-(k, \lambda)$ for a sufficiently small k is expressed as a positive or negative acute angle plus $n\pi$, its limit $\eta_-(0, \lambda)$ is equal to $n\pi$, where n is an integer. This means that $\eta_-(0, \lambda)$ changes discontinuously. When $A(0, \lambda) = 0$, the limit $\eta_-(0, \lambda)$ of the phase shift $\eta_-(k, \lambda)$ is equal to $(n + 1/2)\pi$. It is not important for our discussion except for $A(0, 1) = 0$, which we call the critical case and will discuss later.

Second, for a sufficiently small k, if $A(E, \lambda)$ decreases as λ increases, $\eta_-(k, \lambda)$ increases monotonically. Assume that in the variant process $A(E, \lambda)$ may decrease through the value zero, but does not stop at this value. As $A(E, \lambda)$ decreases, each time tan $\eta_-(k, \lambda)$ for the sufficiently small k changes sign from positive to negative, $\eta_-(0, \lambda)$ jumps by π. However, each time tan $\eta_-(k, \lambda)$ changes sign from negative to positive, $\eta_-(0, \lambda)$ remains invariant. Conversely, if $A(E, \lambda)$ increases as λ increases, $\eta_-(k, \lambda)$ decreases monotonically. As $A(E, \lambda)$ increases, each time tan $\eta_-(k, \lambda)$ changes sign from negative to positive, $\eta_-(0, \lambda)$ jumps by $-\pi$, and each time tan $\eta_-(k, \lambda)$ changes sign from positive to negative, $\eta_-(0, \lambda)$ remains invariant.

Third, as λ increases from zero to one, $V(x, \lambda)$ changes from zero to the given potential $V(x)$ continuously. Each time $A(0, \lambda)$ decreases from near and larger than the value zero to smaller than that value, the denominator in Eq. (27) changes sign from positive to negative and the remaining factor remains positive, such that the phase shift at zero momentum $\eta_-(0, \lambda)$ jumps by π. Conversely, each time $A(0, \lambda)$ increases across the value zero, the phase shift at zero momentum $\eta_-(0, \lambda)$ jumps by $-\pi$. Each time $A(0, \lambda)$ decreases from near and larger than the value x_0^{-1} to smaller than that value, the numerator in Eq. (27) changes sign from positive to negative, but the remaining factor remains negative, such that the phase shift at zero momentum $\eta_-(0, \lambda)$ does not jump. Conversely, each time $A(0, \lambda)$ increases across the value x_0^{-1}, the phase shift at zero momentum $\eta_-(0, \lambda)$ does not jump either.

Therefore, the phase shift $\eta_-(0)/\pi$ is just equal to the times $A(0, \lambda)$ decreases across the value zero as λ increases from zero to one, subtracted by the times $A(0, \lambda)$ increases across that value. As discussed in the previous section, we have proved that the difference of the two times is nothing but the number of bound states n_-, namely, for the noncritical cases, the Levinson theorem for the one-dimensional Schrödinger equation in the odd-parity case is

$$\eta_-(0) = n_-\pi \tag{29}$$

Fourth, we now discuss the critical case where the logarithmic derivative $A(0, 1)$ ($\lambda = 1$) is equal to zero. In the critical case, the constant solution $\psi(x) = c$ ($c \neq 0$) in the range $[x_0, \infty)$ for zero energy will match this $A(0, 1)$ at x_0. In the critical case, it is obvious that there exists a half-bound state for both the even-parity case and the odd-parity case. A half-bound state is not a bound state, because its wave function is finite, but not square-integrable. As λ increases from a number near and less than one and finally reaches one, if the logarithmic derivative $A(0, \lambda)$ decreases and finally reaches, but not cross, the value zero, according to the discussion in the previous section, a scattering state becomes a half-bound state when $\lambda = 1$. On the other hand, the denominator in Eq. (27) is proportional to k^2 such that $\tan \eta_{-}(k, 1)$ tends to infinity. Namely, the phase shift $\eta_{-}(0, 1)$ jumps by $\pi/2$. Therefore, for the critical case the Levinson theorem becomes

$$\eta_{-}(0) - \pi/2 = n_{-}\pi \tag{30}$$

Conversely, as λ increases and reaches one, if the logarithmic derivative $A(0, \lambda)$ increases and finally reaches the value zero, a bound state becomes a half-bound state when $\lambda = 1$, and the phase shift $\eta_{-}(0, 1)$ jumps by $-\pi/2$. In this situation, the Levinson theorem (30) still holds.

Finally, for the even-parity case, the only change is to replace the phase shift $\eta_{-}(0)$ with the phase shift $\eta_{+}(0) + \pi/2$. Therefore, the Levinson theorem for the one-dimensional Schrödinger equation in the even-parity case is

$$\begin{aligned} \eta_{+}(0) + \pi/2 &= n_{+}\pi \qquad \text{for the noncritical case} \\ \eta_{+}(0) &= n_{+}\pi \qquad \text{for the critical case} \end{aligned} \tag{31}$$

Note that for the free particle in the even-parity case, there is a half-bound state at $E = 0$. It is the critical case where $\eta_{+}(0) = 0$ and $n_{+} = 0$. Combining Eqs. (29)–(31), we obtain the Levinson theorem for the one-dimensional Schrödinger equation as Eq. (5).

5. DISCUSSION

Now we discuss the general case where the potential $V(x)$ has a tail at $x \geq x_0$. First, we assume that

$$V(x) = bx^{-2}, \qquad x \geq x_0 \tag{32}$$

It is obvious that when $b < -1/4$ there is an infinite number of bound states for the Schrödinger equation (8) such that the Levinson theorem (5) is violated. When $b \geq -1/4$, let

$$j(j + 1) = b, \qquad j = -1/2 + (b + 1/4)^{1/2} \geq -1/2 \tag{33}$$

The Schrödinger equation (8) becomes the same as the radial equation in three dimensions except that the phase shift is $\eta_{-}(k, \lambda) - j\pi/2$ now. Repeating the proof in our previous paper [6], we obtain the modified Levinson theorem for the Schrödinger equation (8) with the potential (32) in the noncritical case:

$$\eta_-(0) - j\pi/2 = n_-\pi, \qquad \eta_+(0) + (1 - j)\pi/2 = n_+\pi \tag{34}$$

In other words, the Levinson theorem (5) is violated. It is obvious that the Levinson theorem will be violated more seriously if the potential tail decays at infinity slower than the potential tail (32). On the other hand, if the potential tail decays at infinity faster than the potential tail (32), for an arbitrarily given small positive number ϵ, there always exists a larger enough number x_0 such that

$$(-\epsilon)(-\epsilon + 1)x^{-2} < V(x) < \epsilon(\epsilon + 1)x^{-2}, \qquad x \geq x_0 \tag{35}$$

Since ϵ is arbitrarily small, no modification is needed to the Levinson theorem (5).

In conclusion, we establish the one-dimensional Levinson theorem (5) for the Schrödinger equation in one dimension with the potential satisfying

$$V(-x) = V(x), \qquad \lim_{x\to\infty} x^2 V(x) = 0 \tag{36}$$

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